Generalization of the Nambu-Goldstone theorem

Yoshimasa Hidaka

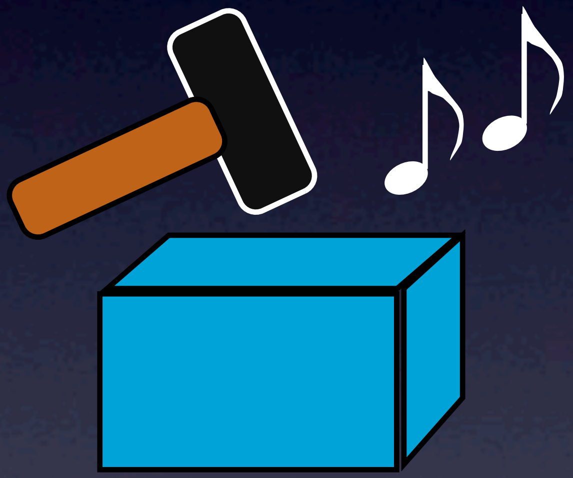
(RIKEN)

Based on Phys. Rev. Lett. 110, 091601 (2013), 1203.1494 [hep-th]

Zero modes in nature

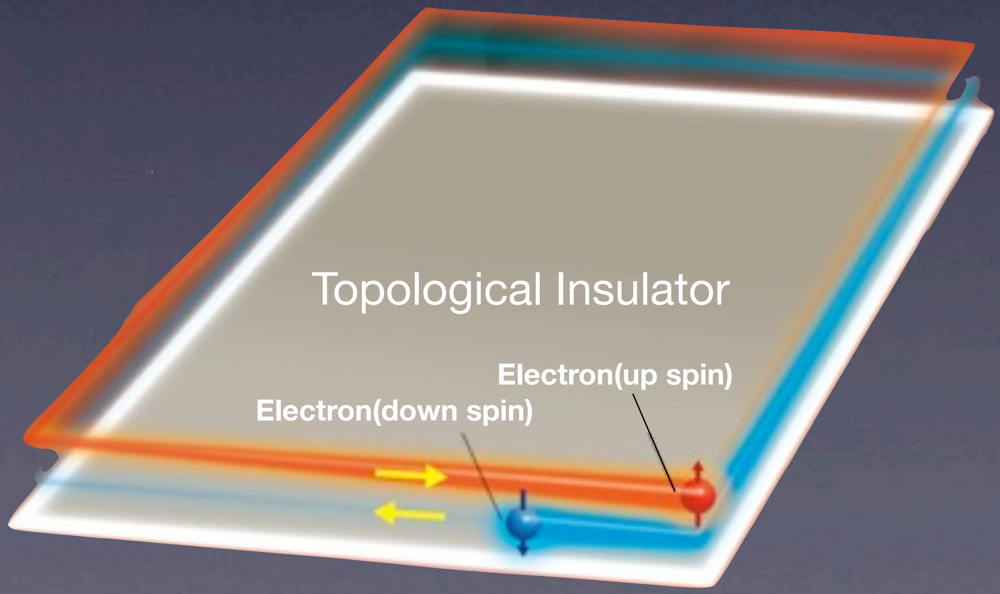
Light (Photon)

Gauge symmetry



Crystal Vibrations (Phonon)

Spontaneous symmetry breaking of translation



Edge modes in topological insulator

Topological

Spontaneous Symmetry Breaking

the Universe:

Particle-antiparticle

Gauge symmetry

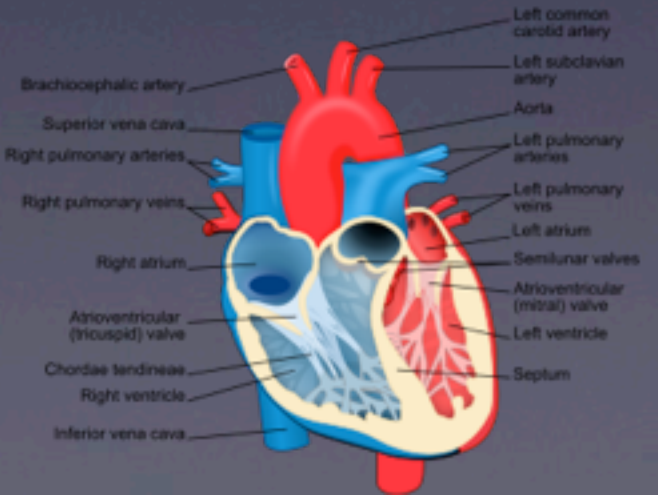
Chiral symmetry...

More familiar cases:

Handedness

Traffic (left or right hand)

Heart



Nambu-Goldstone theorem

Spontaneous breaking
of continuum symmetry

⇒ gapless mode (NG mode)

- Relation between broken symmetries and NG modes?
- Dispersion of NG mode?

It is well studied in each case.

QCD, superfluid, ferromagnet,...

Nambu-Goldstone theorem
(Lorentz invariant system)

Nambu('60), Goldstone(61), Nambu Jona-Lasinio('61), Goldstone, Salam, Weinberg('62).

$$N_{NG} = N_{BS}$$

# of NG modes # of broken symmetries

Dispersion relation

$$E_k = |\boldsymbol{k}|$$

NG modes in QCD

Pion

SSB of chiral symmetry

$$SU(2)_L \times SU(2)_R \to SU(2)_V$$

$$N_{\rm BS} = 3, \quad N_{\rm NG} = 3$$

Dispersion: $E = |\boldsymbol{k}|$ **Type-I**

NG modes in Kaon condensed CFL phase

Miransky, Shovkovy ('02) Schafer, Son, Stephanov, Toublan, and Verbaarschot ('01)

$$SU(2)_I \times U(1)_Y \to U(1)_{\rm em}$$

$$N_{\rm BS} = 3, \quad N_{\rm NG} = 2$$

Dispersion: $E \propto \boldsymbol{k}^2$ **Type-II**

Example of NG mode in nonrelativistic systems

SSB of space-time symm.

Phonon in crystal

translation, rotation, Galilei

$$N_{\mathrm{BS}} = 9, \quad N_{\mathrm{NG}} = 3$$

SSB of internal symm.

Spin waves in ferromagnet

SSB of rotation O(3)→O(2)

$$N_{\mathrm{BS}} = 2, \quad N_{\mathrm{NG}} = 1$$

Generalization

Nielsen - Chadha ('76)

$$N_{\text{type-I}} + 2N_{\text{type-II}} \geq N_{\text{BS}}$$

Type-I: $E_k \propto |\boldsymbol{k}|^{2n+1}$ **Type-II:** $E_k \propto |\boldsymbol{k}|^{2n}$

Schafer, Son, Stephanov, Toublan, and Verbaarschot ('01)

$$\langle [Q_a, Q_b] \rangle = 0 \;\Rightarrow\; N_{\text{NG}} = N_{\text{BS}}$$

Watanabe - Brauner ('11)

$$N_{\text{BS}} - N_{\text{NG}} \leq \frac{1}{2} \text{rank} \langle [Q_a, Q_b] \rangle$$

Example of Type-II modes

	N_{BS}	$N_{\text{type-I}}$	$N_{\text{tyep-II}}$	$\frac{1}{2}\text{rank}\langle[Q_a, Q_b]\rangle$	$N_{\text{type-I}} + 2N_{\text{type-II}}$
Spin wave in ferromanget O(3)→O(2)	2	0	1	1	2
NG modes in Kaon condensed CFL $SU(2) \times SU(1)_Y \to U(1)_{em}$	3	1	1	1	3
Kelvin waves in vortex superfluid translation P_x, P_y	2	0	1	1	2

Known examples satisfy

$$N_{\text{type-I}} + 2N_{\text{type-II}} = N_{\text{BS}} \qquad N_{\text{BS}} - N_{\text{NG}} = \frac{1}{2}\text{rank}\langle[Q_a, Q_b]\rangle$$

Recent development

Watanabe, Murayama ('12), YH ('12)

- $N_{\rm BS} - N_{\rm NG} = \frac{1}{2}{\rm rank}\langle[Q_a, Q_b]\rangle$
- $N_{\text{type-I}} + 2N_{\text{type-II}} = N_{\rm BS}$
- $N_{\text{type-II}} = \frac{1}{2}{\rm rank}\langle[Q_a, Q_b]\rangle$

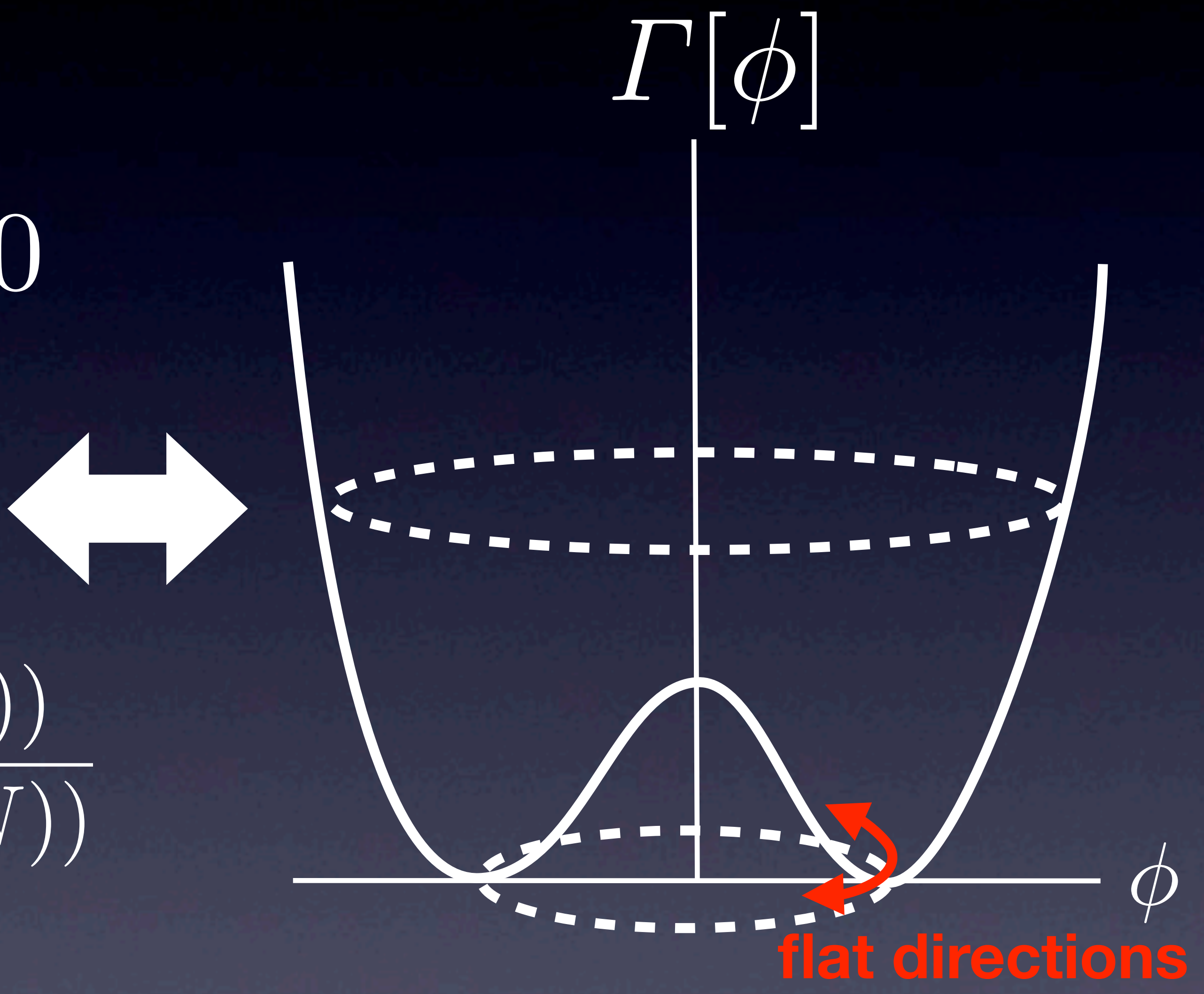
Spontaneous Symmetry breaking

$$\langle [\phi_i, Q_a] \rangle \equiv \mathrm{tr}\rho\, [\phi_i, Q_a] \neq 0$$

$$a = 1, \cdots, N_{\mathrm{BS}}$$

Vacuum: $\rho = |\Omega\rangle\langle\Omega|$

In medium: $\rho = \dfrac{\exp(-\beta(H - \mu N))}{\mathrm{tr}\exp(-\beta(H - \mu N))}$



Suppose the classical action is invariant under

$$\phi_i \to \phi_i + \epsilon^a [Q_a, \phi_i]$$

Effective action $\Gamma[\phi]$ satisfies

$$\int d^d x \frac{\delta \Gamma[\phi]}{\delta \phi_i(x)} \langle [Q_a, \phi_i(x)] \rangle = 0$$

$$\Rightarrow \int d^d x \frac{\delta^2 \Gamma[\phi]}{\delta \phi_j(y) \delta \phi_i(x)} \langle [Q_a, \phi_i(x)] \rangle = 0$$

Inverse of propagator $D^{-1}_{ji}(y, x) = \dfrac{\delta^2 \Gamma[\phi]}{\phi_j(y) \delta \phi_i(x)}$

has a zero mode.

# of independent zero modes

= # of independent eigenvectors

(elastic variables)

For Lorentz invariant system

Goldstone, Salam, Weinberg ('62)

$$\langle [Q_a, \phi_i(x)] \rangle \equiv M_i^{(a)}$$

➡ $$D_{ji}^{-1}(p^2 = 0) M_i^{(a)} = 0$$

$$N_{\mathrm{NG}} = N_{\mathrm{BS}}$$

In general, N_{BS} ≠ num. of independent eigenvectors

$\langle [Q_a, \phi_i(x)] \rangle$ is not always the eigenvector.

Ex.: String

Low - Manohar's argument

Low, and Manohar ('02)

order parameter: $\langle \phi(x) \rangle$

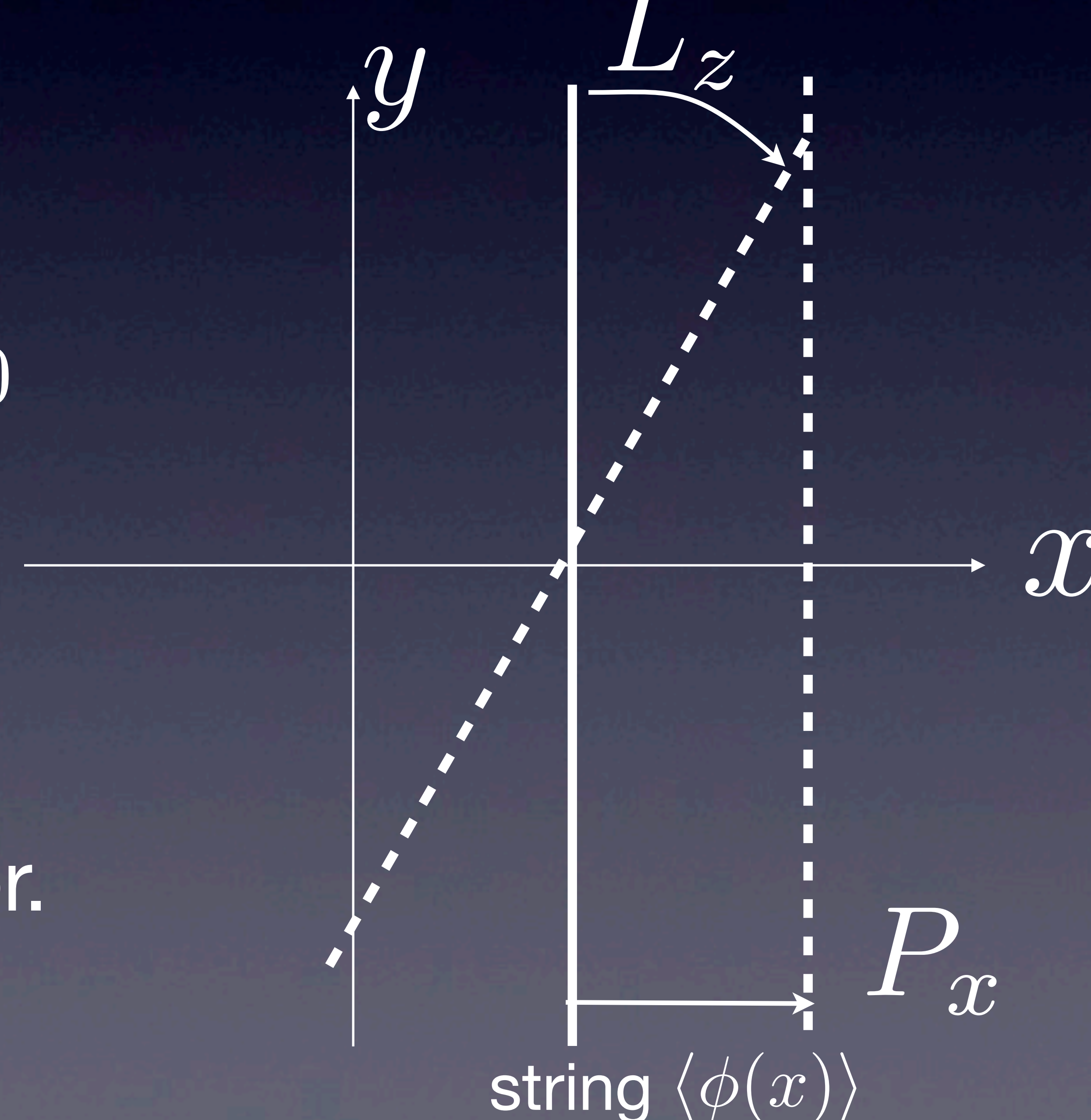
trans.: $\langle [P_x, \phi] \rangle = i\partial_x \langle \phi \rangle \neq 0$

rot.: $\langle [L_z, \phi] \rangle = -iy\partial_x \langle \phi \rangle \neq 0$

Two broken symm.,

but one elastic variable.

Rotation is not the eigenvector.

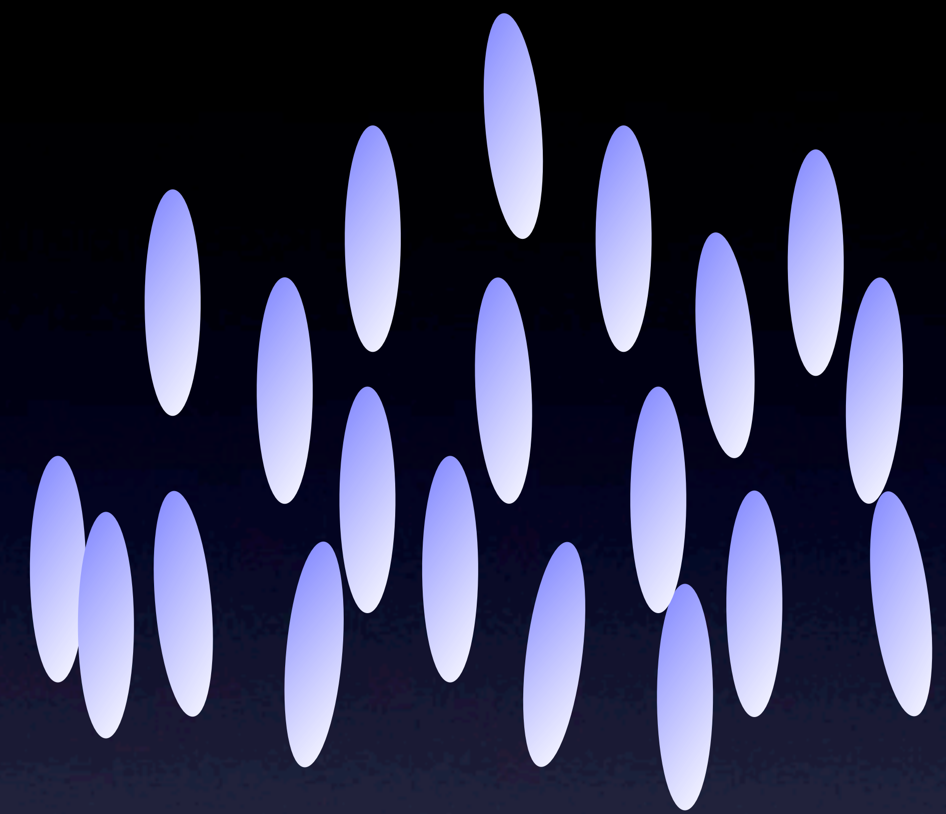


Nontrivial Ex.: nematic liquid crystal

Nematic phase

spatial rotation O(3)→O(2)

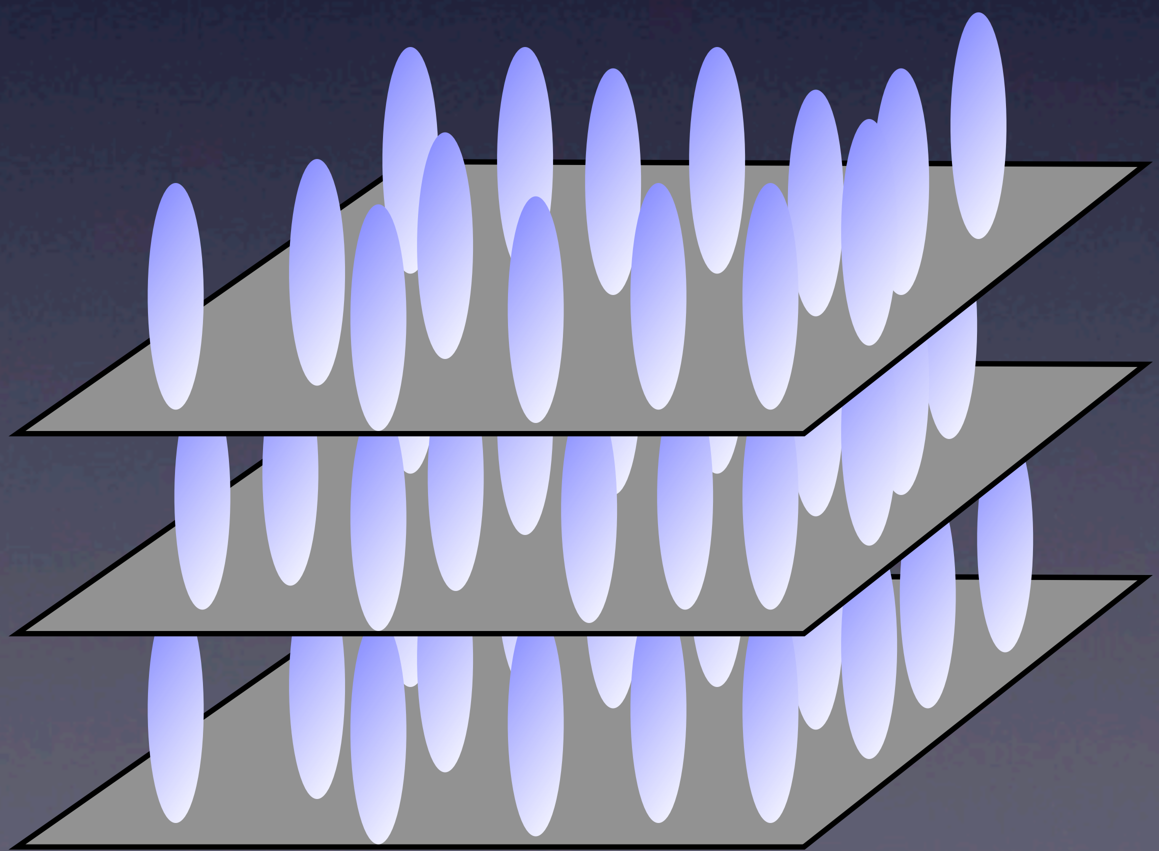
Two broken symm.
Two elastic variables



Smectic A phase

spatial rotation O(3)→O(2)
SSB of the translation.

Three broken symm.
One elastic variable



of independent NG modes is not always # of elastic variables

Ex.: spin wave in ferromagnet

Spin symm.: O(3)→ O(2)

Broken symm.: S_x, S_y

Two eigenvectors: $\epsilon^{ijz}\langle s_z \rangle$

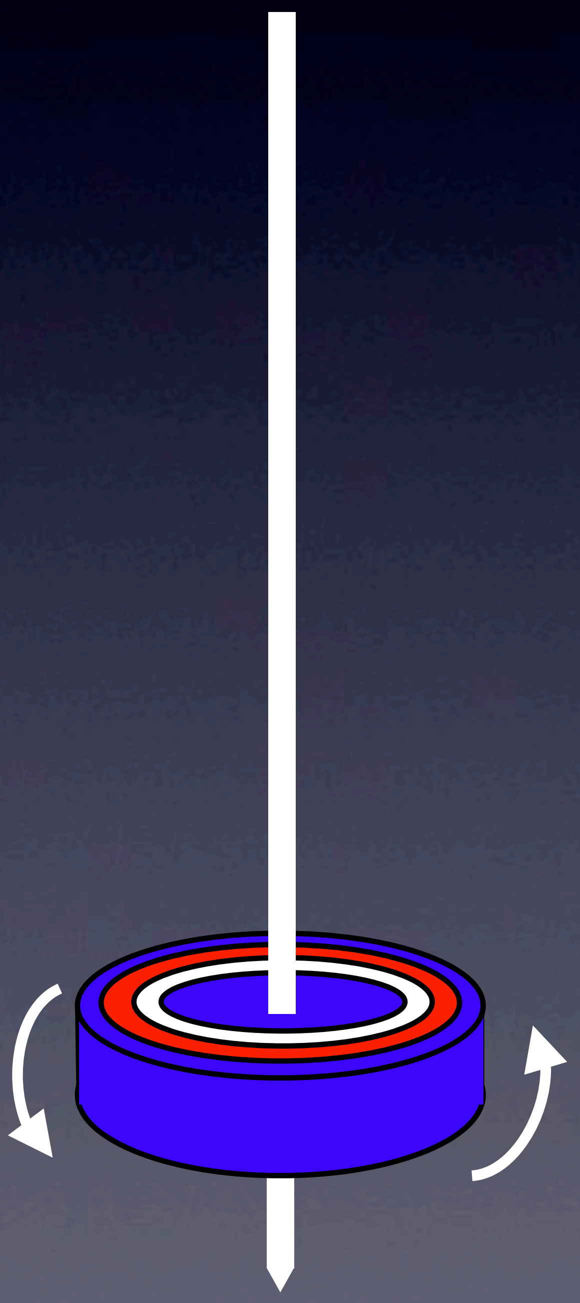
Two independent elastic variables

One spin wave

Intuitive example for type-II NG modes

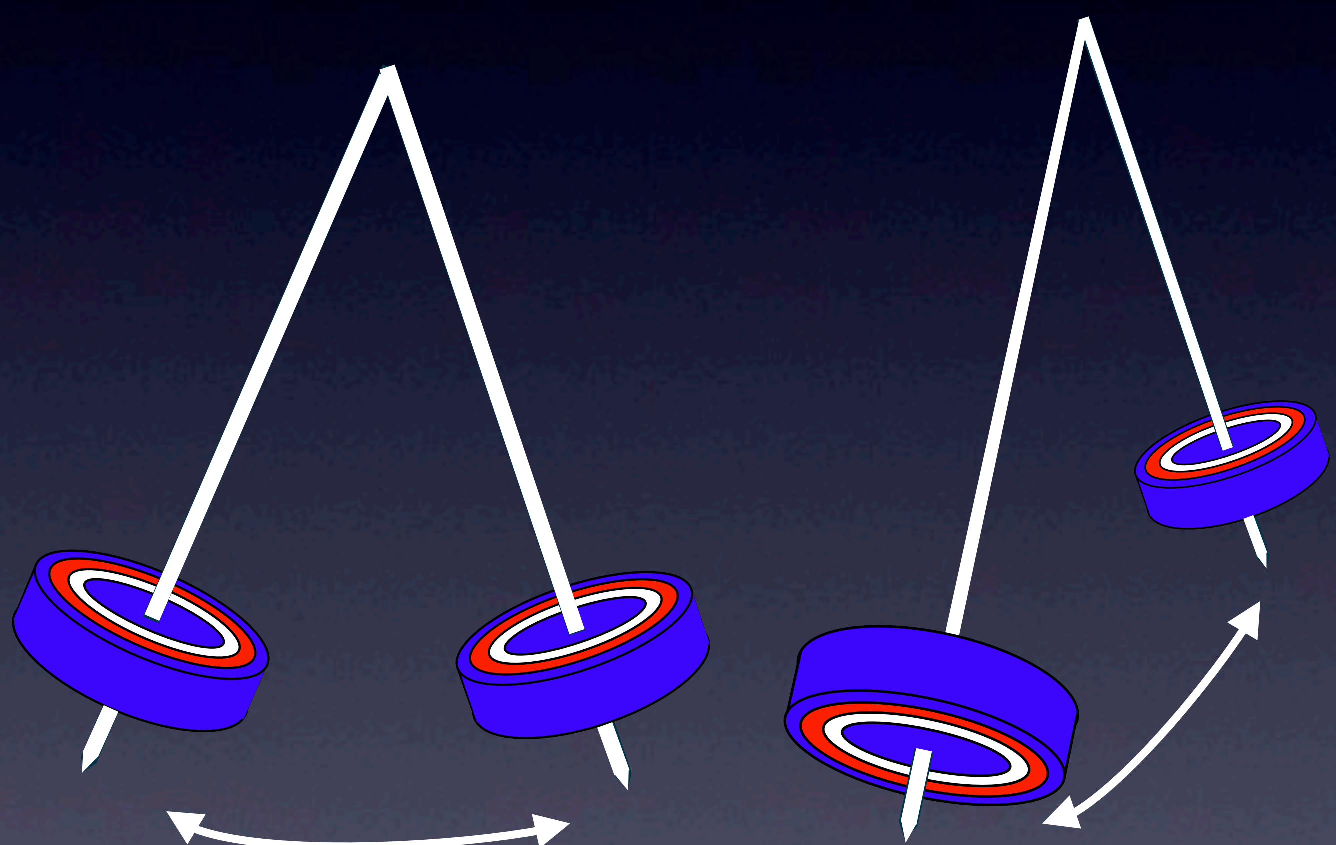
Pendulum with a spinning top

- Rotation symmetry is explicitly broken by a weak gravity
- Rotation along with z axis is unbroken.
- Rotation along with x or y is broken.
- The number of broken symmetry is two.



Intuitive example for type-II NG modes

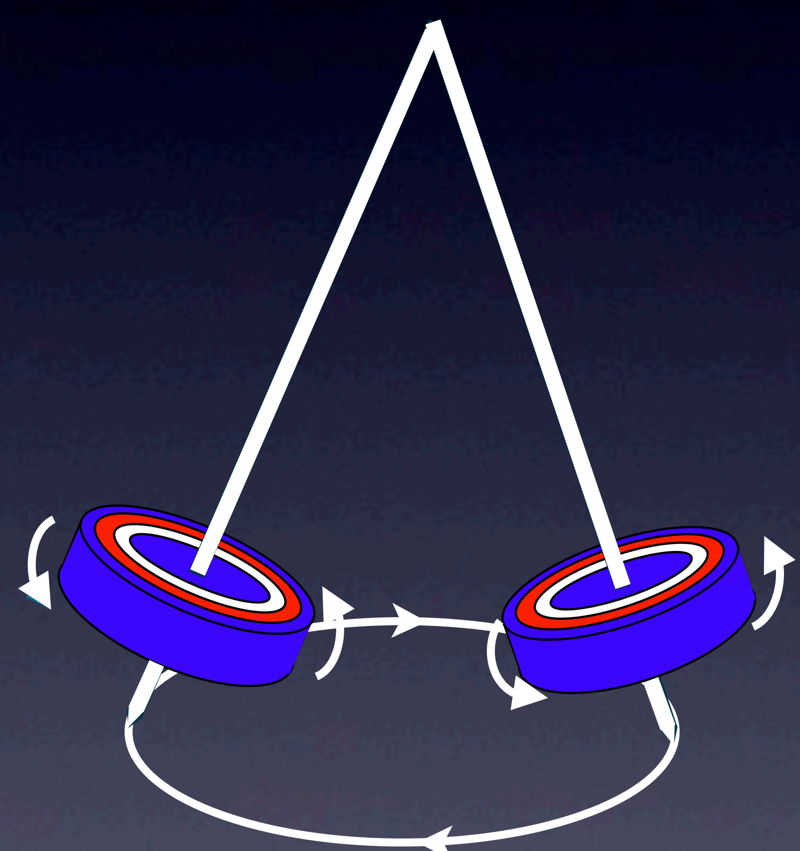
Pendulum has two oscillation motions



if the top is not spinning.

Intuitive example for type-II NG modes

If the top is spinning,



the only one rotation motion (Precession) exists.

In this case, $\{L_x, L_y\}_P = L_z \neq 0$

The number of modes decreases if the angular momentum is nonzero.

What is a generalization?

Correspondence

Classical Hamiltonian formalism		Projection operator method
$\{A_n, A_m\}_P$	⟺	$-i\langle [A_n, A_m] \rangle$
$H(A_n)$		$\Gamma(A_n)$

Justification: projection operator method

Mori ('65)

Spontaneous symmetry breaking

$$\langle [Q_a, \phi_i(x)] \rangle \equiv M_i^{(a)}$$

Q and ϕ : canonical pair

$$\langle [Q_a, Q_b] \rangle = 0$$

Canonically independent (Type-I)

$$\langle [Q_a, Q_b] \rangle \neq 0$$

Not canonically independent (Type-II)

cf. Nambu ('04)

The number of Type-II pairs

$$N_{\text{type-II}} = \frac{1}{2}\text{rank}\langle[Q_a, Q_b]\rangle$$

The number of Type-I pairs

$$N_{\text{type-I}} = N_{\text{BS}} - 2N_{\text{type-II}}$$

- $N_{\text{type-I}} + 2N_{\text{type-II}} = N_{\text{BS}}$
- $N_{\text{BS}} - N_{\text{NG}} = \frac{1}{2}\text{rank}\langle[Q_a, Q_b]\rangle$
- $N_{\text{type-II}} = \frac{1}{2}\text{rank}\langle[Q_a, Q_b]\rangle$

Dispersion relation

Simple Hamiltonian system

$$\{\phi, n\}_P = 1$$

$$H = \frac{a(k)}{2} n^2 + \frac{b(k)}{2} \phi^2$$

$$\partial_t \phi = \{\phi, H\}_P = a(k) n$$

$$\partial_t n = \{n, H\}_P = -b(k) \phi$$

$$\partial_t^2 \phi + a(k) b(k) \phi = 0$$

$$\partial_t^2 \phi + a(k)b(k)\phi = 0$$

$$a(k) = a_0 + a_2 k^2$$

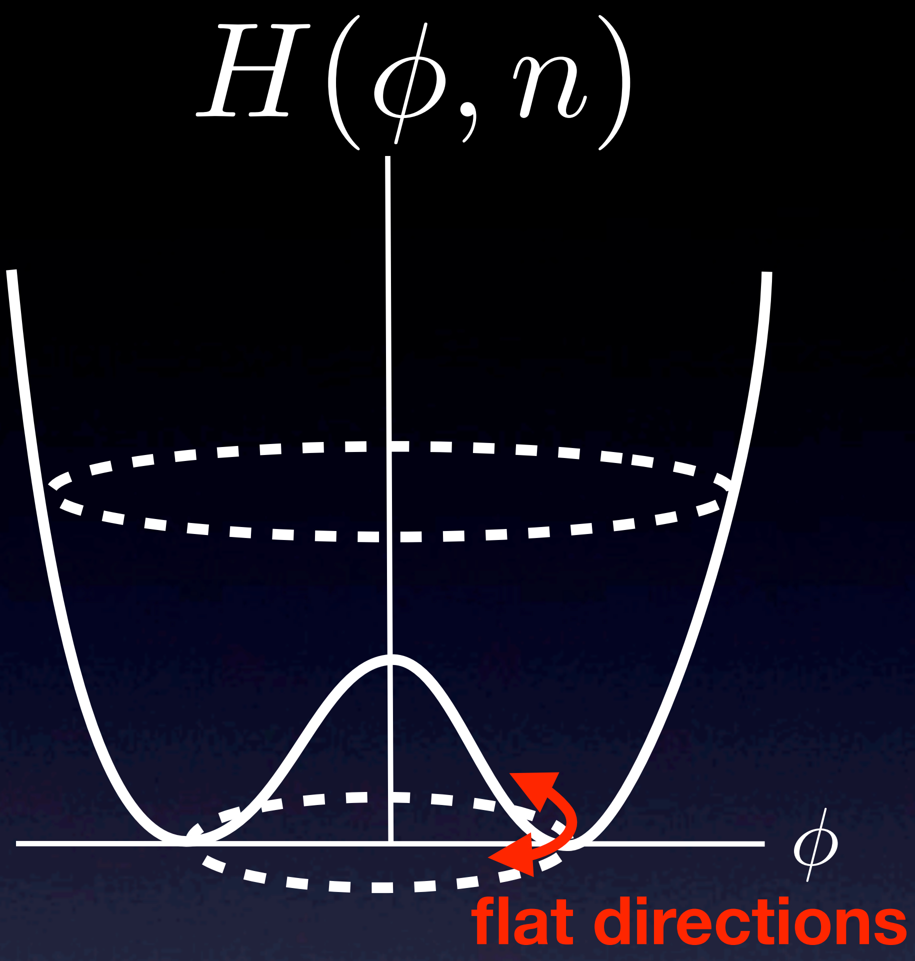
$$b(k) = b_0 + b_2 k^2$$

$$a(k)b(k) = a_0 b_0 + (a_0 b_2 + a_2 b_0)k^2 + a_2 b_2 k^4$$

	a_0	b_0	エネルギー
gapped	nonzero	nonzero	$\sqrt{a_0 b_0}$
gapless (type-I)	zero	nonzero	$\sim \lvert\boldsymbol{k}\rvert$
gapless (type-II)	zero	zero	$\sim \boldsymbol{k}^2$

cf. Nambu ('04)

$$H = \frac{a(k)}{2} n^2 + \frac{b(k)}{2} \phi^2$$



Type-I

$$\langle [Q, \phi] \rangle \neq 0, \langle [Q, n] \rangle = 0$$

$$a(k) \simeq a_0, \ \ b(k) \simeq b_2 k^2 \Rightarrow E_k = \sqrt{a_0 b_2} k$$

Type-II

$$\langle [Q_1, n_2] \rangle \neq 0, \langle [Q_2, n_1] \rangle = 0 \text{ with } n = n_2 \ \ \phi = n_1$$

$$a(k) \simeq a_2 k^2, \ \ b(k) \simeq b_2 k^2 \Rightarrow E_k = \sqrt{a_2 b_2} k^2$$

Formalism

- What are canonical pairs?
- What corresponds to Poisson bracket?
- What is Hamiltonian?

Projection operator method

Mori ('65)

$$\partial_t A_n(t,\boldsymbol{k}) = i[H, A_n(t,\boldsymbol{k})]$$

⬇

Generalized Langevin Eq.

$$\partial_t A_n(t,\boldsymbol{k}) = M_{nm}(\boldsymbol{k})\mathit{\Gamma}^{ml}(\boldsymbol{k})A_m(t,\boldsymbol{k})$$

Streaming term

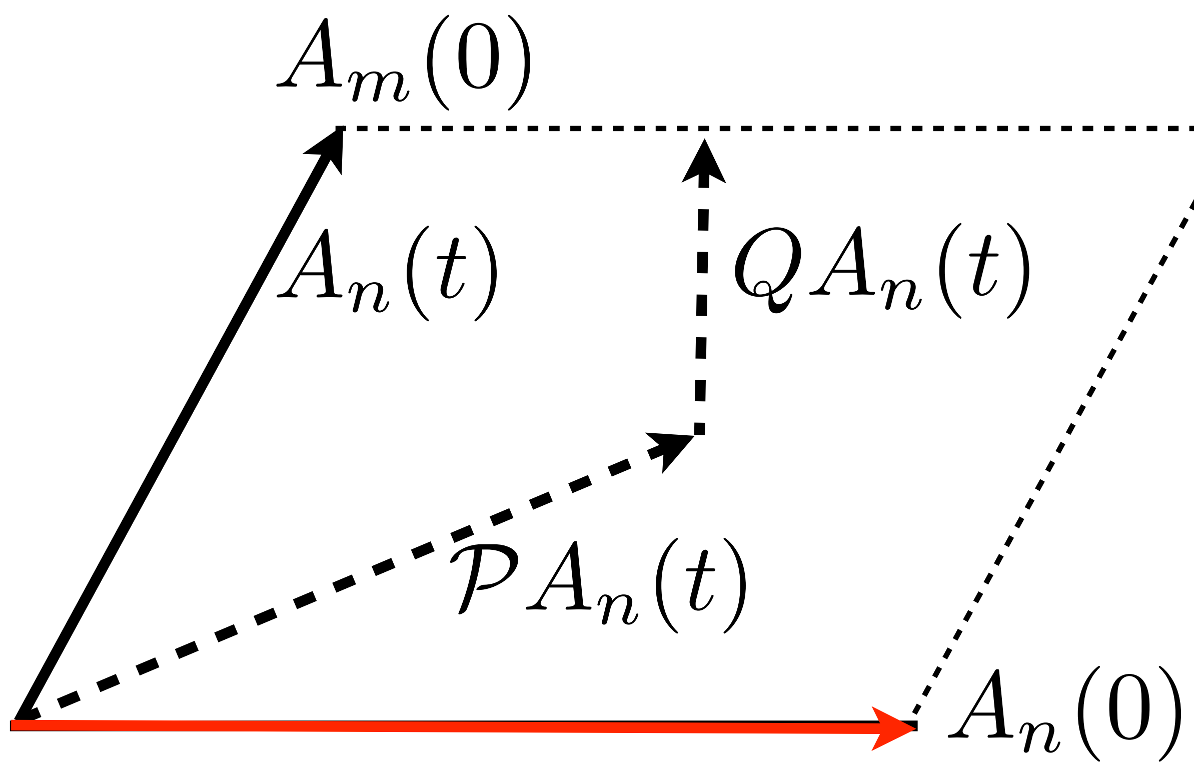
$$-\int_0^\infty ds K_{nm}(t-s,\boldsymbol{k})\mathit{\Gamma}^{ml}(\boldsymbol{k})A_l(s,\boldsymbol{k}) + R_n(t,\boldsymbol{k})$$

Dissipation term **Noise**

Projection operator method

Mori ('65)

A set of operators $\{A_n\}$



Projection operator method

Mori ('65)

Expectation value

$$\langle \mathcal{O} \rangle \equiv \frac{\mathrm{tr} e^{-\beta H} \mathcal{O}}{\mathrm{tr} e^{-\beta H}}$$

Inner product

$$(\mathcal{O}_1, \mathcal{O}_2) \equiv \frac{1}{\beta} \int_0^\beta d\tau \langle e^{\tau H} \mathcal{O}_1 e^{-\tau H} \mathcal{O}_2^\dagger \rangle$$

Metric

$$g_{nm}(\boldsymbol{x} - \boldsymbol{y}) \equiv (A_n(0, \boldsymbol{x}), A_m(0, \boldsymbol{y}))$$

Covariant vector

$$A^n(t, \boldsymbol{x}) \equiv \int d^3y g^{nm}(\boldsymbol{x} - \boldsymbol{y}) A_m(t, \boldsymbol{y})$$

Projection operator method

Mori ('65)

Projection operator

$$PB(t,\boldsymbol{x}) \equiv \int d^3y A_n(0,\boldsymbol{y})(B(t,\boldsymbol{x}), A^n(0,\boldsymbol{y}))$$

Streaming term

$$M_{nm}(t,\boldsymbol{x}) \equiv -i\langle [A_n(0,\boldsymbol{x}), A_m^\dagger(0,\mathbf{0})]\rangle$$

Memory function

$$K_{nm}(t,\boldsymbol{x}) \equiv \beta\theta(t)(R_n(t,\boldsymbol{x}), R_m(0,\mathbf{0}))$$

Noise term

$$R_n(t,\boldsymbol{x}) \equiv e^{itQ\mathcal{L}} Qi\mathcal{L} A_n(0,\boldsymbol{x}) \quad \text{with } Q = 1 - P$$

Projection operator gives

$$(z\delta_n{}^l - (M_{nk}(\boldsymbol{k}) - K_{nk}(z,\boldsymbol{k}))\Gamma^{kl}(\boldsymbol{k}))G_l{}^m(z,\boldsymbol{k}) = \delta_n{}^m$$

in Laplace-Fourier space

where $G_n{}^m(t,\boldsymbol{x}) \equiv (A_n(t,\boldsymbol{x}), A^m(0,\boldsymbol{0}))$

Correspondence

Classical Hamiltonian system		Quantum system
$\{\phi_n, \phi_m\}_P$	⟺	$-i\langle[\phi_n, \phi_m]\rangle$
$H(\phi_n)$		$\Gamma(\phi_n)$

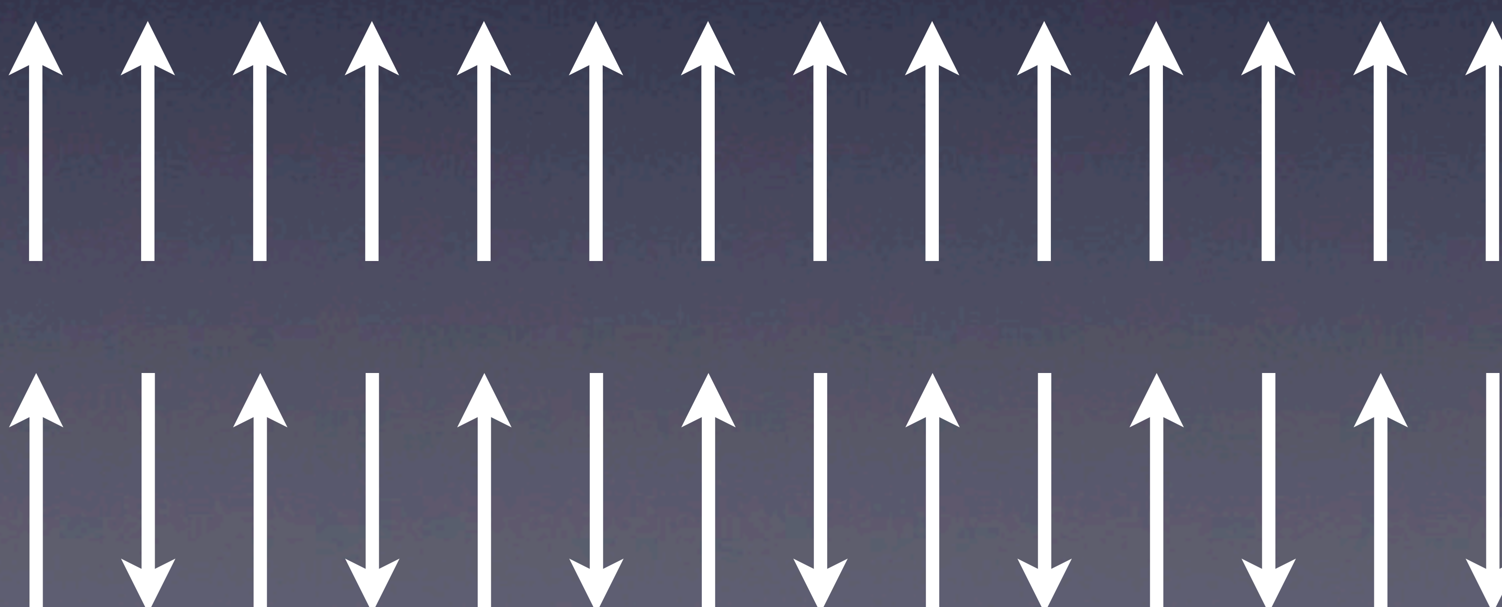
Ex.: Ferro and antiferro magnet

Spin: $s_i(n)\quad i = x, y, z$

Symm.: $O(3)$

charge: $Q_i = \sum_n s_i(n)$

current algebra: $[Q_i, s_j(n)] = i\epsilon_{ijk} s_k(n)$



Ferro $\langle s_z(n) \rangle = m$

Antiferro $\langle s_z(n) \rangle = (-1)^n m$

NG mode in Ferromagnet

Commutation relation: $$-i\langle[Q_x, Q_y]\rangle = \langle Q_z\rangle = mV$$

NG field: $$s_i(k) \equiv \sum_n s_i(n)e^{ik\cdot n} \quad i = x, y$$

Susceptibility: $$\chi(k) \equiv \frac{1}{TV}(s_i(k), s_i(k)) \simeq \frac{1}{a\boldsymbol{k}^2}$$

Canonical pair: $$-i\frac{1}{V}\langle[s_x(k), s_y(k)]\rangle \simeq m$$

Equation of motion:

$$\partial_t s_i(k) = am\boldsymbol{k}^2\epsilon_{ij}s_j(k)$$

$\Rightarrow$ $E_k = \pm am\boldsymbol{k}^2$ Type-II

NG mode in antiferromagnet

Commutation relation:

$$-i\langle[Q_x, Q_y]\rangle = \langle Q_z\rangle = 0$$

NG field: $\phi_i(k) \equiv \sum_n (-1)^n s_i(n) e^{ik\cdot n}$ $\quad i = x, y$

Susceptibilities:

$$\chi_s(k) \equiv \frac{1}{TV}(s_i(k), s_i(k)) \simeq \frac{1}{a}$$

$$\chi_\phi(k) \equiv \frac{1}{TV}(\phi_i(k), \phi_i(k)) \simeq \frac{1}{bk^2}$$

Canonical pair: $-i\frac{1}{V}\langle[s_i(k), \phi_j(k)]\rangle \simeq \epsilon_{ij} m$

Equation of motion:

$$\partial_t \phi_i(k) = ma\epsilon_{ij} s_j(k) \qquad \partial_t s_i(k) = \epsilon_{ij} mbk^2 \phi_j(k)$$

➡ $E_k = \pm\sqrt{ab}k$ Type-I

Nambu-Goldstone Theorem

Spontaneous symmetry breaking

$$-i\langle[Q_a, \phi_i(t, \boldsymbol{x})]\rangle \equiv [M_{n\phi}]_{ai},$$

$$\det M_{n\phi} \neq 0$$ ➡ **SSB**

$Q_a = \int d^3x n_a(t, \boldsymbol{x})$ **:Broken charges**

$\phi_i(t, \boldsymbol{x})$ **:would-be NG field**

Slow variables

$$A_m = \{\delta n_a(t, \boldsymbol{x}), \delta\phi_i(t, \boldsymbol{x})\}$$

At $\boldsymbol{k} = \boldsymbol{0}$

$$\partial_0 \begin{pmatrix} \phi \\ n \end{pmatrix} = \begin{pmatrix} 0 & M_{\phi n} \\ M_{n\phi} & M_{nn} \end{pmatrix} \begin{pmatrix} \Gamma^{\phi\phi} & \Gamma^{\phi n} \\ \Gamma^{n\phi} & \Gamma^{nn} \end{pmatrix} \begin{pmatrix} \phi \\ n \end{pmatrix}$$

$$= \begin{pmatrix} M_{\phi n}\Gamma^{n\phi} & M_{\phi n}\Gamma^{nn} \\ M_{n\phi}\Gamma^{\phi\phi} + M_{nn}\Gamma^{n\phi} & M_{n\phi}\Gamma^{\phi n} + M_{nn}\Gamma^{nn} \end{pmatrix} \begin{pmatrix} \phi \\ n \end{pmatrix}$$

$$= \begin{pmatrix} -F^{-1}G & F^{-1} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \phi \\ n \end{pmatrix}$$

where $F^{-1} = M_{\phi n}\Gamma^{nn}$ and $G = M_{nn}M_{\phi n}^{-1}$

$$N_{\text{massive}} = \frac{1}{2}\text{rank}(F^{-1}G) = \frac{1}{2}\text{rank}(M_{nn})$$

$$N_{\text{BS}} - N_{\text{NG}} = N_{\text{massive}} = \frac{1}{2}\text{rank}(\langle [Q_a, Q_b] \rangle)$$

Watanabe-Brauner

$$\partial_0 \begin{pmatrix} \phi \\ n \end{pmatrix} = \begin{pmatrix} -F^{-1}G & F^{-1} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \phi \\ n \end{pmatrix}$$

$$\Rightarrow \partial_0^2 \phi = F^{-1} G \partial_0 \phi$$

Type-I NG modes: $F^{-1} G \phi_I = 0$

$$\partial_0^2 \phi_I = 0 \quad N_{\text{type-I}} = N_{\text{BS}} - \text{rank}(M_{nn})$$

Type-II NG modes: others

$$G \partial_0 \phi_{II} = 0 \quad N_{\text{type-II}} = \frac{1}{2} \text{rank}(M_{nn})$$

$$\Rightarrow N_{\text{type-I}} + 2 N_{\text{type-II}} = N_{\text{BS}}$$

Nielsen-Chadha

At finite temperature

$$\{A_n, A_m\}_P = M_{nm}$$

$$\Downarrow$$

$$\{A_n, A_m\}_T \equiv M_{nm} - L_{nm}$$

Onsager coefficient

$$L_{nm}(\boldsymbol{k}) = \int_0^\infty dt \int_0^\beta d\tau \int d^3x e^{-i\boldsymbol{x}\cdot\boldsymbol{k}} \langle R_n(t - i\tau, \boldsymbol{x}) R_m(0, \boldsymbol{0}) \rangle$$

$$R_n(t, \boldsymbol{x}) \equiv e^{itQ\mathcal{L}} Q i \mathcal{L} A_n(0, \boldsymbol{x})$$

$$\mathcal{L} A_n \equiv [H, A_n]$$

Finite temperature

At $\boldsymbol{k} = \mathbf{0}$

$$\partial_0 \begin{pmatrix} \phi \\ n \end{pmatrix} = \begin{pmatrix} L_{\phi\phi} & M_{\phi n} \\ M_{n\phi} & M_{nn} \end{pmatrix} \begin{pmatrix} \mathit{\Gamma}^{\phi\phi} & \mathit{\Gamma}^{\phi n} \\ \mathit{\Gamma}^{n\phi} & \mathit{\Gamma}^{nn} \end{pmatrix} \begin{pmatrix} \phi \\ n \end{pmatrix}$$

$$= \begin{pmatrix} -\tilde{F}^{-1}G & \tilde{F}^{-1} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \phi \\ n \end{pmatrix}$$

where $\tilde{F}^{-1} = (M_{\phi n} - L_{\phi\phi}(M_{n\phi})^{-1}M_{nn})\mathit{\Gamma}^{nn}$ and $G = M_{nn}M_{\phi n}^{-1}$

$$N_{\text{massive}} = \frac{1}{2}\text{rank}(\tilde{F}^{-1}G) = \frac{1}{2}\text{rank}(M_{nn})$$

$$N_{\text{BS}} - N_{\text{NG}} = N_{\text{massive}} = \frac{1}{2}\text{rank}(\langle[Q_a, Q_b]\rangle)$$

Effective Lagrangian approach

Leutwyler('94)   Watanabe, Murayama ('12)

Write down all possible term

$$\mathcal{L} = \frac{1}{2}\rho_{ab}\pi^a\dot{\pi}^b + \frac{\bar{g}_{ab}}{2}\dot{\pi}^a\dot{\pi}^b - \frac{g_{ab}}{2}\partial_i\pi^a\partial_i\pi^b$$

+higher

No Lorentz symmetry:

The first derivative term may appear.

Lagrangian is invariant under symmetry transformation up to surface term.

$$\Rightarrow \rho_{ab} \propto -i\langle[Q_a, j_b^0(x)]\rangle$$

Watanabe, Murayama ('12)

Summary

For internal symmetry

- **Independent elastic variable=*N*$_{\mathrm{BS}}$**
- $N_{\mathrm{BS}} - N_{\mathrm{NG}} = \frac{1}{2}\mathrm{rank}\langle[Q_a, Q_b]\rangle$
- $N_{\text{type-I}} + 2N_{\text{type-II}} = N_{\mathrm{BS}}$
- $N_{\text{type-II}} = \frac{1}{2}\mathrm{rank}\langle[Q_a, Q_b]\rangle$

Summary

SSB of space-time symmetry

- Num. of Elastic variables≦NBS

- How many NG modes exist?

cf. Watanabe and Murayama, 1302.4800

1. $$\frac{\partial}{\partial t}e^{i\mathcal{L}t} = e^{i\mathcal{L}t}i\mathcal{L} = e^{i\mathcal{L}t}\mathcal{P}i\mathcal{L} + e^{i\mathcal{L}t}\mathcal{Q}i\mathcal{L}$$

2. $$\begin{aligned}\frac{1}{z-i\mathcal{L}} &= \frac{1}{z-i\mathcal{L}}(z-\mathcal{Q}i\mathcal{L})\frac{1}{z-\mathcal{Q}i\mathcal{L}} \\ &= \frac{1}{z-i\mathcal{L}}(z-i\mathcal{L}+\mathcal{P}i\mathcal{L})\frac{1}{z-\mathcal{Q}i\mathcal{L}} \\ &= \frac{1}{z-\mathcal{Q}i\mathcal{L}} + \frac{1}{z-i\mathcal{L}}\mathcal{P}i\mathcal{L}\frac{1}{z-\mathcal{Q}i\mathcal{L}}\end{aligned}$$

$$\Rightarrow \quad e^{i\mathcal{L}t} = e^{\mathcal{Q}i\mathcal{L}t} + \int_0^t ds e^{i\mathcal{L}(t-s)} Pi\mathcal{L}e^{\mathcal{Q}i\mathcal{L}s}$$

1. and 2. ⇓

$$\frac{\partial}{\partial t}e^{i\mathcal{L}t} = e^{i\mathcal{L}t}\mathcal{P}i\mathcal{L} + \int_0^t ds e^{i\mathcal{L}(t-s)}\mathcal{P}i\mathcal{L}e^{\mathcal{Q}i\mathcal{L}s}\mathcal{Q}i\mathcal{L} + e^{\mathcal{Q}i\mathcal{L}t}\mathcal{Q}i\mathcal{L}.$$

Multiplying $A_n(0)$ ⇒ **Generalized Langevin Eq.**